In this installment we derive a central result in compressible fluid flow showing the inner working of the converging-diverging nozzle. The key relations are the jump conditions are the mass conservation equation

\[ \rho\_1 V\_1 A\_1 = \rho\_2 V\_2 A\_2 \; \]

and the energy equation

\[ h\_1 + \frac{V\_1^2}{2} = h\_2 + \frac{V\_2^2}{2} \; .\]

As a reminder, the enthalpy is defined in terms of the specific internal energy $e$ by $h = e + P/\rho$. Correspondingly, the first law of thermodynamics (as usually expressed in terms of the specific internal energy)

\[ de = T ds + \frac{P}{\rho^2} d\rho \]

becomes

\[ dh = T ds + \frac{dP}{\rho} \; .\]

The final set of equations come from the specific fluid that is flowing. For simplicity, we will deal with compressible gas flow and will take the gas to be ideal. The equation of state for an ideal gas is

\[ \frac{P}{\rho} = RT \]

with the usual relations between the heat capacity at constant pressure $c\_P$ and the heat capacity at constant volume $c\_V$ of

\[ c\_P – c\_V = R \; \]

and

\[ c\_P/c\_V = \gamma \; , \]

where $\gamma$ is a constant usually taken to have the value of 1.4 for ordinary air.

For an ideal gas, the specific enthalpy (like the specific energy) is strictly a function of temperature given by

\[ dh = c\_P dT \; , \]

where we will often assume that $c\_P$ is constant over a wide range of conditions; an assumption that allows us to integrate to $h = c\_P T$ immediately. In other words, we assume the gas is a calorically perfect ideal gas – this is actually a reasonably good assumption given that [$c\_P$ for nitrogen](https://www.engineeringtoolbox.com/nitrogen-d_977.html) varies only about 1.5% over the range from 300-500K and only about 5.5% when the range extends to 700 K.

The speed of sound in the ideal gas is also only a function of temperature given by

\[ c = \sqrt{ \gamma R T } \; .\]

The final assumption is that the flow is isentropic, which is a good assumption since the flow happens quickly so that a negligible amount of heat flows into the system. Under this assumption, the equation of state can be substituted into the first law to yield

\[ c\_P dT = \frac{dP}{\rho} \; .\]

The next step is to eliminating the density by way of the ideal gas law to get an equation in terms of pressure and temperature

\[ c\_P dT = \frac{dP}{P/RT} \; . \]

Re-arranging gives a simple set of differential equations

\[ \frac{dT}{T} = \frac{R}{c\_P} \frac{dP}{P} \; ,\]

which integrates to

\[ \frac{T\_1}{T\_2} = \left( \frac{P\_1}{P\_2} \right)^{\frac{\gamma-1}{\gamma}} \; ,\]

where $R = c\_P - c\_V$ and $c\_P/c\_V = \gamma$.

When the analogous steps are followed with $P$ eliminated in favor of $\rho$, one arrives at

\[ \frac{T\_1}{T\_2} = \left( \frac{\rho\_1}{\rho\_2} \right)^{\gamma – 1} \; . \]

The value of this latter equation is that it allows the elimination of temperature (simply equation both forms of $T\_1/T\_2$ and the emergence of the polytropic equation

\[ \frac{P\_1}{P\_2} = \left( \frac{\rho\_1}{\rho\_2} \right)^{\gamma} \; \]

relating pressure and density for an isentropic flow.

With these assumptions and corresponding relations in hand, we now use them to relate the conditions from the inlet of the gas from a reservoir (e.g. a fuel tank) to the outlet of the nozzle. The rocket engine will be the prototype for this situation. We need to be able to relate the thermodynamics at each point in the flow to the thermodynamics in the reservoir. Since the gas in the reservoir is not moving with respect to the nozzle we introduce the concept of stagnation, in which the flow is envisioned to have come to a stop and the only energy present being stored as internal energy. The third jump condition relating enthalpy and bulk flow velocity at different points will be important.

Since there is a qualitative change between subsonic and supersonic flow (see last post’s discussion of [how the area is related to Mach number](http://underthehood.blogwyrm.com/?p=1393)), we will need to stitch together the two flows (stagnant-to-sonic with sonic-to-supersonic) at the critical point where $M = 1$. Variables associated with this point are called critical and are decorated with a star (e.g. $A^\*, P^\*$ and so on). In addition, we will decorate all variables associated with the reservoir where the flow is stagnant with a 0 subscript (e.g. $A\_0, P\_0$ and so on).

Let’s start on the subsonic side by relating the thermodynamics in the tank to the corresponding variables in the flow as it moves in the converging portion of the nozzle. Since the flow is stagnant in the tank, we find that the stagnation enthalpy $h\_0$ relates to the flow anywhere else in the nozzle by

\[ h\_0 = h\_1 + \frac{V\_1^2}{2} \; . \]

Since the gas is calorically perfect, the enthalpy can be eliminated in favor of the temperatures to arrive at

\[ c\_P T\_0 = c\_P T\_1 + \frac{V\_1^2}{2} \; .\]

Dividing by $c\_P T\_1$ and using the fact that $T\_1 = {c\_1}^2/\gamma R$ gives

\[ \frac{T\_0}{T\_1} = 1 + \frac{V\_1^2}{2 c\_P T\_1} = 1 + \frac{V\_1^2}{2 c\_P ({c\_1}^2/\gamma R)} \; .\]

In this last relation, the ratio of the flow velocity to the speed of sound is the Mach number. Dropping the ‘1’ subscript and using the relations between $c\_P$, $c\_V$, and $R$, we arrive at what we will call the stagnation temperature relation

\[ \frac{T\_0}{T} = 1 + \frac{(\gamma-1)}{2} M^2 \;. \]

Using the relationship between the temperature and pressure derived above gives the corresponding stagnation pressure equation

\[ \frac{P\_0}{P} = \left[ 1 + \frac{(\gamma-1)}{2} M^2 \right]^{\frac{\gamma}{\gamma - 1}} \; \]

Return to the mass flow equation and, first eliminate the density using the equation of state and multiply by $c/c$, then express the sound speed in terms of the temperature to get

\[ {\dot m} = \rho A V = \frac{P}{R T} A \frac{V}{c} c = P A M \frac{c}{RT} = P A M \sqrt{ \frac{\gamma}{R T}} \; .\]

Next, solve the stagnation pressure equation for $P$

\[ P = P\_0 \left( \frac{T\_0}{T} \right)^{\frac{\gamma}{1 - \gamma}} \]

and then substitute into the mass flow equation

\[ {\dot m} = P\_0 \left( \frac{T\_0}{T} \right)^{\frac{\gamma}{1 - \gamma}} A M \sqrt{ \frac{\gamma}{R T}} \left( \frac{T\_0}{T\_0} \right)^{1/2} \\ = P\_0 \sqrt{\frac{\gamma}{R T\_0}} \left(\frac{T\_0}{T}\right)^{\frac{\gamma}{1-\gamma}+\frac{1}{2}} A M \; .\]

Simplifying and using the stagnation temperature equation gives

\[ {\dot m} = P\_0 \sqrt{\frac{\gamma}{RT\_0}} A M \left[ 1 + \frac{\gamma-1}{2} M^2 \right]^{\frac{\gamma+1}{2(1-\gamma)}} \; .\]

This combination is constant no matter where in the flow it is evaluated. Equate the expression evaluated at the critical point to the expression anywhere else in the flow to get

\[ P\_0 \sqrt{\frac{\gamma}{RT\_0}} A^\* \left[ 1 + \frac{\gamma-1}{2} \right]^{\frac{\gamma+1}{2(1-\gamma)}} = P\_0 \sqrt{\frac{\gamma}{RT\_0}} A M \left[ 1 + \frac{\gamma-1}{2} M^2 \right]^{\frac{\gamma+1}{2(1-\gamma)}} \; .\]

Eliminating the terms common to both sides and rearranging we can express the cross-sectional area at any point in the nozzle to the point where the flow goes critical as

\[ \frac{A}{A\*} = \frac{1}{M} \left(1+\frac{\gamma-1}{2}\right)^{\frac{\gamma+1}{2(1-\gamma)}} \left[ 1+ \frac{\gamma-1}{2} M^2 \right]^{\frac{\gamma+1}{2(\gamma-1)}} \; .\]

This is the well-known relation that forms the basis for designing and analyzing the nozzle flow. It will be the basis for our final post on compressible flow next month.