This month’s column, the second in a series of three on compressible flow, derives the key relationships for understanding steady, isentropic flow in a converging-diverging nozzle. The starting point of these derivations are the three isentropic jump conditions derived in the last post:

\[ \begin{array}{ll} {\textrm{continuity:}}& {\rho}\_1 V\_1 A\_1 = {\rho}\_2 V\_2 A\_2 \equiv {\dot m}\\{\textrm{momentum:}}& (\rho\_1 V\_1^2 + P\_1) A\_1 = (\rho\_2 V\_2^2 + P\_2) A\_2 \\{\textrm{energy:}}& h\_1 + \frac{{V\_1}^2}{2} = h\_2 + \frac{{V\_2}^2}{2} \end{array} \; , \]

along with the [first law of thermodynamics](https://en.wikipedia.org/wiki/First_law_of_thermodynamics) (cast in intensive variables by dividing both sides of $$dE = TdS – P dV$$ by mass $$m = \rho V$$)

\[ de = T ds + \frac{P}{\rho^2} d \rho = \frac{P}{\rho^2} d \rho \; ,\]

and the [equation of state for an ideal gas](https://en.wikipedia.org/wiki/Ideal_gas) (also in intensive variables)

\[ \frac{P}{\rho} = R T \; . \]

The $$T ds$$ term can be dropped since isentropic flow means that the specific entropy remains constant. Note that the specific internal energy and specific enthalpy ($$h = e + P/\rho$$) are simply functions of temperature given by $$de = c\_{{\mathcal V}} dT$$ and $$dh = c\_P dT$$, respectively, where $$c\_{{\mathcal V}} and $$c\_P$$ are the heat capacities at constant volume and constant pressure and are taken to be constants. Other thermodynamic relations will be taken as given since proof will take us too far afield.

There are three ingredients in understanding the [steady, isentropic flow through a converging-diverging nozzle](https://en.wikipedia.org/wiki/Isentropic_nozzle_flow) (also known as a [de Laval nozzle](https://en.wikipedia.org/wiki/De_Laval_nozzle)): 1) calculating the Mach number, 2) relating the cross-sectional area of the nozzle ($$A$$) to the Mach number, and 3) relating the thermodynamics variables of temperature ($$T$$), the pressure ($$P$$) and the density ($$\rho$$) to the Mach number. This post will deal with items 1 and 2 with item 3 being covered in the next post.

This post and the next are a synthesis and clarification of the ideas and materials found in Chapter 9 of Merle Potter’s *Fluid Mechanics Demystified* and the 5-part series of YouTube lectures on compressible fluid flow from the University of Florida ([part 1](https://www.youtube.com/watch?v=CnAEwDQxeKo), [part 2](https://www.youtube.com/watch?v=ZzyU7BHCllY), [part 3](https://www.youtube.com/watch?v=o-UGasX1atc), [part 4](https://www.youtube.com/watch?v=MdYYgyHNDFY), and [part 5](https://www.youtube.com/watch?v=L-6gvVuiAQ4))

# Calculating the Mach Number

The central parameter in the theory is the Mach number, which is innocently defined as the ratio of the flow speed to the speed of sound. What makes this definition so deceptively simple looking is we are used to thinking that the speed of sound is constant since the temperature range that we normally encounter is such that the variations in the speed of sound is negligible. However, interesting compressible fluid flows often have very large variations in temperature over short durations or small distances and so Mach number varies depending on the location of the flow within the nozzle.

To calculate the speed of sound, which we will denote as $$c$$, we take the first differential of the continuity and momentum equations and since we are interested in the local speed over a column of fluid with constant cross section we can ignore variations in the area (imagine being outside). The resulting equations are:

\[ d \rho c + \rho d c = 0 \; \]

and

\[ d\rho c^2 + 2 c \rho dc + dP = 0 \; . \]

Solving the first equation for $$\rho dc = - c d \rho$$, substituting this result into the second, and solving for $$c^2$$ gives

\[ c^2 = \frac{d P}{d \rho} \; \]

An ideal gas obeys a polytropic equation of state

\[ P = \rho^{\gamma} \; , \]

where the adiabatic index $$\gamma$$ is defined as

\[ \gamma = \frac{c\_P}{c\_{{\mathcal V}}} \; .\]

As a result,

\[ \frac{dP}{d\rho} = \gamma \rho^{\gamma - 1} = \gamma \frac{\rho^{\gamma-1}}{\rho} = \gamma \frac{P}{\rho} \; .\]

Substituting this result back into the first equation and taking the square root gives

\[ c = \sqrt{ \frac{\gamma P }{\rho} } = \sqrt{\gamma R T} \; , \]

where the ideal gas law was used in the last step.

The Mach number is the ratio of the flow speed to the speed of sound and is given by

\[ M = \frac{V}{c} \; . \]

# Relating Cross-Sectional Area and the Mach Number

Most of us have had the experience with running water where we restrict the cross-sectional area of the outlet and watch the fluid flow increase (a thumb over the end of a garden hose being the usual scenario). The reason for this speed increase is that the pressure difference between the inlet and the outlet sets up a given mass flow rate. When the area drops, the flow speed has to increase to maintain that flow rate.

This approach works for flows in most everyday experiences but fails for compressible flow, explaining why a simple converging nozzle is insufficient to force a fluid into supersonic flow.

To see why this is we will derive a relationship between the cross-sectional area and the Mach number.

Start by taking the first differential of the third jump condition

\[ d \left( h + \frac{V^2}{2} \right) = dh + V dV = 0 \; .\]

Next, eliminate the differential of the enthalpy using the first law ($$dh = dP/\rho$) to get

\[ \frac{dP}{\rho + V dV = 0 \; .\]

Multiply the first term by $$d\rho/d\rho$$ to get

\[ \frac{dP d\rho}{\rho d\rho} + VdV = 0 \; .\]

Recognizing that $$c^2 = dP/d\rho$$ and dividing both sides by $$V^2$$ gives

\[ \frac{c^2}{V^2} \frac{d\rho}{\rho} + \frac{dV}{V} = 0 \;. \]

Rearranging this equation gives

\[ \frac{d\rho}{\rho} = -M^2 \frac{dV}{V} \; . \]

Next take the differential of the first jump condition and divide by $$\rho V A$$ to get

\[ \frac{d\rho}{\rho} + \frac{dV}{V} + \frac{dA}{A} = 0 \;. \]

Finally substitute into this equation the previous one and re-arrange to arrive at

\[ \frac{dV}{V} \left( M^2 – 1 \right) = \frac{dA}{A} \; . \]

This equation contains a wealth of information:

* When $$M > 1$$ the flow slows down (speeds up) when the area decreases (increases)
* When $$M < 1$$ the flow speeds up (slows down) when the area decreases (increases)
* When $$M = 1$$ the flow remains at the same speed even as the area changes

These observations explain why a simple converging nozzle can’t force the flow to supersonic speeds. Once the pressure difference between the inlet and outlet is large enough that the flow speed equals the speed of sound ($$M=1$$), the flow is choked and can get no faster. Practitioners usually speak of this situation is that choked flow can’t communicate a further pressure drop at the outlet upstream so that the flow can respond.

It is also worth pointing out something that is easy to overlook in the above manipulation. While the steps may seem haphazard and uninspired there is a method to the madness. There is a hierarchy of thermodynamic relations that are combined: the first law of thermodynamics is a universal law; the third jump condition represents energy conservation for a general fluid flow; and the equation of state for an ideal gas is a specific law for a very special kind of fluid. The first law links a fluid’s energy conservation with the speed of sound for an ideal fluid, linking flow velocity to Mach number and the compressibility of the fluid (as represented by $$d\rho$$). The final step uses another universal law – mass continuity – to arrive at a relationship between a fractional change in the flow speed to a fractional change in volume (as represented by $$dA$$).

Next post puts these pieces together with a set of relations between the fluid’s thermodynamic variables and the Mach number to explain the function of a converging-diverging nozzle and how it can push fluid flows past the speed of sound.

Relating the Thermodynamic Variables to Mach Number

Now that the Mach number is defined, the next step is to recast the isentropic jump conditions so that the individual thermodynamic variables become solely functions of the Mach number.

To this end, start by introducing the concept of stagnation defined as the thermodynamic point where the fluid has zero flow speed. A subscript of ‘0’ will decorate each thermodynamic variable to remind us that its value corresponds to this case.